OPTIMAL AND SUBOPTIMAL GUIDANCE AND CONTROL FOR LOW THRUST ORBITAL TRANSFER

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ABSTRACT

A suboptimal feedback solution to the problem of guidance and control of a vehicle during flight is presented. Optimum open loop solutions for the minimum time and minimum fuel problem for low thrust orbital transfer have been previously obtained. As a further step in the implementation of the resulting trajectories, it is desired to obtain closed loop suboptimal controllers which generate these trajectories. To this end, the nonlinear equations of motion for orbital transfer are modeled by a linear time varying system of assumed form. This system is assumed stationary over subintervals of time which allows an on-board computer to generate a sequential control law which minimizes an integral of a quadratic form of error and control effort in such a fashion as to produce system trajectories which do not differ significantly from the optimal trajectories. Computation time and storage requirements are such as to suggest feasibility of the proposed method for on-line operation.

INTRODUCTION

Much recent attention has been given to the solution of optimal control problems for nonlinear aerospace systems. This effort has resulted in a variety of methods for the computational solution of nonlinear two point boundary-value problems. [1,2] In these boundary-value problems half of the boundary conditions are specified at the final time. This implies that an a-priori knowledge of the complete system dynamics must be known over the time interval of operation $t \in (t_o, t_f)$. Thus in a large number of cases, solution of an optimal nonlinear control problem results in the determination of an open-loop control for a system with known dynamics over the time interval of operation. In many instances, a closed-loop control is desired. Also, if there are process variations, environmental changes, or uncertainties in the system model, the complete knowledge of system dynamics necessary to predetermine the open-loop control cannot be obtained. For the case of a linear system with known constant coefficients, the closed-loop control can be obtained with relative ease. [3] This paper attempts to develop, and provide experimental justification for, a suboptimal guidance and control scheme for low thrust orbital vehicles based, in part, on the identification of a linear model and use of the real time computational simplicity of linear systems with quadratic cost functions.

Specifically this paper develops a method for on-line control which can be computed rapidly due to the identification of a linear model for the plant, and which introduces feedback by sequentially monitoring the system at discrete time instants and updating the control. The model chosen is a linear time-varying system which is assumed stationary over subintervals of time, thus allowing a controller to generate a sequential control law which minimizes, not the given performance index, but a closely related one. The resulting control is of course only an approximation to the optimum control. However, due to process and environmental variations which cannot be foreseen, it may substantially reduce the cost over that which results from a predetermined open-loop control.

DERIVATION OF CONTROL LAW

In the problem to be examined, it is desired to force the system to follow a precomputed "optimum" solution even though the system may encounter certain environmental changes or noise which were not accounted for in the precomputed solution. In general, additional cost is required for doing this as well as large computer storage requirements. Several schemes have previously been developed for accomplishing this. [4,5] Here the emphasis is placed on the simplicity of the control, the ease, and speed with which it may be computed, and possible minimization of storage requirements.

Assume a given system of the form (1) with cost function (2).

$$\dot{\underline{x}} = f\,(\underline{x}, \underline{\mu}, t) \qquad \underline{x}\,(t_o) = \underline{x}_o \tag{1}$$

$$J = \theta\,[\tilde{\underline{x}}\,(t_f)] + \int_{t_o}^{t_f} \Phi\,(\tilde{\underline{x}}\,\underline{\mu},\ t)\ dt \tag{2}$$

Here, $\tilde{\underline{x}} = \underline{x} - \underline{x}_d$, and $\underline{x}_d$ represents the desired final state of the system at time t.

The optimum solution is found using any one of several methods. Then the state vector for $t = t_o, t_1, \ldots, t_{m-1}, t_f$ is stored for future use and designated $\underline{x}_d\,(t_i)$. If the time increments $\Delta t_i = t_{i+1} - t_i$ are equal, $m\Delta t_o = t_f - t_o = m\Delta t_i$. For the suboptimal control scheme the system is identified as

$$\dot{\underline{x}} = A\,(t_i)\,\underline{x} + B\,(t_i)\,\underline{\mu} \tag{3}$$

where A and B are constant matrices over the interval $t \epsilon\,(t_i,\ t_{i+1})$. For each subinterval, a cost function of the form

$$V_i = \frac{1}{2}\,[\underline{x}\,(t_{i+1}) - \underline{x}_d\,(t_{i+1})]'\ P\ [\underline{x}\,(t_{i+1}) - \underline{x}_d\,(t_{i+1})] + \frac{1}{2}\int_{t_i}^{t_{i+1}} \underline{\mu}'\ R\,(t_i)\ \underline{\mu}\ dt \tag{4}$$

is chosen. The elements of P determine how closely the predetermined "optimum" solution should be followed. The total cost is given by

$$\Xi = \sum_{i=o}^{N-1} V_i \tag{5}$$

At each $t = t_i$ the two point boundary-value problem must be solved, at least for the initial values of the control. If the subinterval length is sufficiently small, the control can be held constant over the entire subinterval with only very slight difference from the results with a variable control.

By utilizing the maximum principle of Pontryagin [6], the suboptimal control can be found. The Hamiltonian, H, is written as,

$$H = \frac{1}{2} \underline{\mu}^T R \underline{\mu} + \underline{\lambda}^T [A (t_i) \underline{x} + B (t_i) \underline{\mu}] \tag{6}$$

For notational simplicity it is desirable to write $A(t_i)$ and $B(t_i)$ as A and B since they are constant matrices over each subinterval. By application of the maximum principle the canonic equations are

$$\underline{\mu} = - R^{-1} B^T \underline{\lambda} \tag{7}$$

$$\dot{\underline{x}} = A \underline{x} + B \underline{\mu} \tag{8}$$

$$\dot{\underline{\lambda}} = - A^T \underline{\lambda} \tag{9}$$

$$\underline{x} (t_i) = \underline{x}_i \tag{10}$$

The transversality conditions at $t = t_{i+1}$ gives

$$\underline{\lambda} (t_{i+1}) = P [\underline{x} (t_{i+1}) - \underline{x}_d (t_{i+1})] \tag{11}$$

It is desirable to convert (7) through (10) into equations for which the control can be computed directly. With this in mind, it is convenient to define an n-r dimension vector $\underline{\sigma}$ as being

$$\underline{\sigma} = \begin{bmatrix} \lambda_{r+1} \\ \vdots \\ \lambda_n \end{bmatrix} \tag{12}$$

By adjoining this $\underline{\sigma}$ vector to the $\underline{\mu}$ vector there results

$$\begin{bmatrix} \underline{\mu} \\ \hline \underline{\sigma} \end{bmatrix} = \left[\begin{array}{c|c} \multicolumn{2}{c}{-R^{-1} B^T} \\ \hline 0 & I \end{array}\right] \underline{\lambda} \tag{13}$$

where I is an (r x r) identity matrix. Let

$$M = \left[\begin{array}{c|c} \multicolumn{2}{c}{-R^{-1} B^T} \\ \hline 0 & I \end{array}\right] \tag{14}$$

$$\begin{bmatrix} \underline{\mu} \\ \hline \underline{\sigma} \end{bmatrix} = \underline{\Gamma} \tag{15}$$

Thus,

$$\underline{\lambda} = M^{-1} \underline{\Gamma} \tag{16}$$

$$\dot{\underline{\lambda}} = M^{-1} \dot{\underline{\Gamma}} \tag{17}$$

Equations (8) and (9) become

$$\dot{\underline{x}} = A\,\underline{x} + C\,\underline{\Gamma} \tag{18}$$

$$\dot{\underline{\Gamma}} = \theta\,\underline{\Gamma} \tag{19}$$

where $\theta = -\,M\,A^T\,M^{-1}$

and C is an n x n matrix defined by

$$C = [B \mid 0].$$

The initial conditions on $\underline{x}$ are given by (11), and the endpoint conditions on $\underline{\Gamma}$ are given by

$$\underline{\Gamma}\,(t_{i+1}) = \left[\begin{array}{c}\underline{\mu}\,(t_{i+1}) \\ \hline \underline{\sigma}\,(t_{i+1})\end{array}\right] = \left[\begin{array}{c|c}-R^{-1}B^T & \\ \hline 0 & I\end{array}\right] P\,[\underline{x}\,(t_{i+1}) - \underline{x}_d\,(t_{i+1})] \tag{20}$$

$$\underline{\Gamma}\,(t_{i+1}) = M\,P\,[\underline{x}\,(t_{i+1}) - \underline{x}_d\,(t_{i+1})] \tag{21}$$

where I is an (r x r) identify matrix.

The solution of (18), (19), (21) and (10) is given by

$$\left[\begin{array}{c}\underline{x}\,(t-t_i) \\ \hline \underline{\Gamma}\,(t-t_i)\end{array}\right] = \epsilon^{\left[\begin{array}{c|c}A & C \\ \hline \theta & \theta\end{array}\right](t-t_i)} \left[\begin{array}{c}\underline{x}\,(t_i) \\ \hline \underline{\Gamma}\,(t_i)\end{array}\right] \tag{22}$$

where $\underline{\Gamma}\,(t_i)$ is yet to be found. To do this, define

$$\epsilon^{\left[\begin{array}{c|c}A & C \\ \hline \theta & \theta\end{array}\right](t-t_i)} = \left[\begin{array}{c|c}\Phi_{xx}\,(t-t_i) & \Phi_{x\Gamma}\,(t-t_i) \\ \hline \Phi_{\Gamma x}\,(t-t_i) & \Phi_{\Gamma\Gamma}\,(t-t_i)\end{array}\right] \tag{23}$$

At $t = t_{i+1}$, $\underline{\Gamma}$ is known in terms of $\underline{x}$. Utilizing this information, $\underline{\Gamma}(t_i)$ is found to be

$$\underline{\Gamma}\,(t_i) = \eta^{-1}\,[\Psi\,\underline{x}\,(t_{i+1}) + P\,x_d\,(t_{i+1})] \tag{24}$$

$$\eta = P\,\Phi_{x\Gamma}\,(t_{i+1}-t_i) - \Phi_{\Gamma\Gamma}\,(t_{i+1}-t_i) \tag{25}$$

$$\Psi = \Phi_{\Gamma x}\,(t_{i+1}-t_i) - P\,\Phi_{xx}\,(t_{i+1}-t_i) \tag{26}$$

Since the first r components of $\underline{\Gamma}$ comprise the control vector, the control can be applied after computing (24).

This method of control is then combined with a suitable scheme for identification to implement the on-line controller. Quasilinearization [7] and invariant imbedding techniques [8,9] have been found effective for on-line identification [10].

LOW THRUST ORBITAL TRANSFER

As an example of this type of control, consider the problem of minimizing the fuel consumption of a low thrust rocket which is to transfer from the orbit to Earth to the orbit of Mars in fixed time. The orbits of Mars and Earth are assumed to be circular and coplaner, and the gravitational attractions of the two planets are neglected. The problem has been previously formulated and solved for the open-loop control assuming an inequality constraint on propellant mass flow, β, or thrust. [11] The normalized dynamics and boundary conditions are given by

$$
\begin{aligned}
\dot{r} &= w && \text{(Radial velocity)} \\
\dot{w} &= \frac{v^2}{r} - \frac{K}{r^2} + \frac{C\beta}{m}\sin\theta && \text{(Radial acceleration)} \\
\dot{v} &= -\frac{wv}{r} + \frac{C\beta}{m}\cos\theta && \text{(Circumferential acceleration)} \\
\dot{m} &= -\beta && \text{(Mass flow)}
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
r(0) &= 1.0 & r(t_f) &= 1.52 \\
w(0) &= 0.0 & w(t_f) &= 0.00 \\
v(0) &= 1.0 & v(t_f) &= 0.81 \\
m(0) &= 1.0 & m(t_f) &\sim \text{open}
\end{aligned}
\tag{28}
$$

with $K = 1.00$

$C = 1.872$

$\beta_{max} = 0.075$

$\beta_{min} = 0.0$

where the final time, t_f, is 3.816 units which corresponds to 222.0 days and θ is the thrust angle measured from the local horizontal. It is desired to minimize the fuel consumed or equivalently the cost function

$$P = -m(t_f)$$

Computational results show that the open-loop control is a bang-off-bang type. However, in actual practice, due to measurement errors, noise, etc., it may not be desirable to apply this as a precalculated open-loop control, especially since near impact the system may require more than the open-loop β_{max} to match the critical endpoints. For this reason, trajectory control seems feasible since it monitors the system and tends to keep it tracking the precalculated trajectory, although possibly at more cost.

The non-linear dynamics given above are modeled as a linear non-stationary system of the form

$$\dot{w} = \frac{v(t_i)}{r(t_i)} v - \frac{K}{r(t_i)} 2 + \frac{C}{m(t_i)} u_1 \tag{29}$$

$$\dot{v} = \frac{-v(t_i)}{r(t_i)} w + \frac{C}{m(t_i)} u_2 \tag{30}$$

with $u_1 = \beta \sin \theta$ (31)

$u_2 = \beta \cos \theta.$

The total time for the flight (222 days) is divided into 37 subintervals $t \epsilon (t_i, t_{i+1})$ where $i = 0, 1 ---, 36$. Thus each subinterval corresponds to 6 days. It is desired to minimize the cost function

$$J = [\frac{1}{2} p_{11} (w - w_d)^2 + \frac{1}{2} p_{22} (v - v_d)^2]|_{t=t_{i+1}} + \frac{1}{2} \int_{t_i}^{t_{i+1}} \alpha(u_1^2 + u_2^2)\, dt \tag{32}$$

over each of these subintervals where

$p_{11} = 1000.0$

$p_{22} = 1000.0$

and $\alpha = 1.0$.

The values for w_d and v_d at each t_{i+1} are taken from the optimal open-loop trajectory.

The Hamiltonian is given by (6) as

$$H = \frac{1}{2} \alpha(u_1^2 + u_2^2) + \lambda_1 [\frac{v(t_i)}{r(t_i)} v - \frac{K}{r(t_i)^2} + \frac{C}{m(t_i)} u_1] + \lambda_2 [\frac{-v(t_i)}{r(t_i)} w + \frac{C}{m(t_i)} u_2] \tag{33}$$

and equations (9), (13), (15), and (19) yield for the canonic equations:

$$\dot{w} = \frac{v(t_i)}{r(t_i)} v - \frac{K}{r(t_i)^2} + \frac{C}{m(t_i)} u_1 \tag{34}$$

$$\dot{v} = \frac{-v(t_i)}{r(t_i)} w + \frac{C}{m(t_i)} u_2 \tag{35}$$

$$\dot{u}_1 = \frac{v(t_i)}{r(t_i)} u_2 \tag{36}$$

$$\dot{u}_2 = \frac{-v(t_i)}{r(t_i)} u_1 \tag{37}$$

REFERENCES

1. Bellman, R. E., and R. E. Kalaba, Quasilinearization and Nonlinear Boundary-Value Problems, New York: American Elsevier Publishing Co., 1965.

2. Kelley, H., "Method of Gradients," in G. Leitman (ed.), Optimization Techniques, New York: Academic Press, 1962.

3. Kalman, R. E., "Theory of Optimal Control and to Calculus of Variations," in R. Bellman (ed.), Mathematical Optimization Techniques, Berkeley, California: University of California Press, 1963.

4. Kovatch, G., "Optimal Guidance and Control Synthesis for Maneuverable Lifting Space Vehicles," Transactions on Ninth Symposium on Ballistic Missile and Space Technology, August, 1964.

5. Breakwell, J. V., J. L. Speyer, and A. E. Bryson, "Optimization and Control of Nonlinear Systems Using the Second Variation," Journal of Society of Industrial and Applied Mathematics on Control, Vol. 1, No. 1, 1963.

6. Pontryagin, L. S., V. G. Boltyanski, R. V. Gamkrelidge, and E. F. Mishchenko, The Mathmetical Theory of Optimal Processes, Interscience Publishers, Inc., New York, 1962.

7. Sage, A.P. and Eisenberg, B. R. "Nonlinear and Nonstationary System Identification Via Quasilinearization and Differential Approximation" Proceedings, Joint Automatic Control Conference, 1966.

8. Sage, A. P., and G. W. Masters, "On-Line Estimation of States and Parameters for Discrete Nonlinear Dynamical Systems," Proceedings of the National Electronics Conference, Vol. 22, 1966.

9. Detchmendy, D. M., and R. Snidhar, "Sequential Estimation of States and Parameters in Noisy Nonlinear Dynamical Systems," ASME Journal of Basic Engineering, Vol. 88, Series D, No. 2, June, 1966.

10. Sage, A. P., and T. W. Ellis, "Sequential Suboptimal Adaptive Control of Nonlinear Systems," Proceedings of the National Electronics Conference, Vol. 22, 1966.

11. Kenneth, P., and G. E. Taylor, "Solution of Variational Problems with Bounded Control Variables by Means of the Generalized Neuton-Raphsor Method," Recent Advances in Optimization Techniques, Edited by A. Lavi, John Wiley & Sons, New York, 1966.

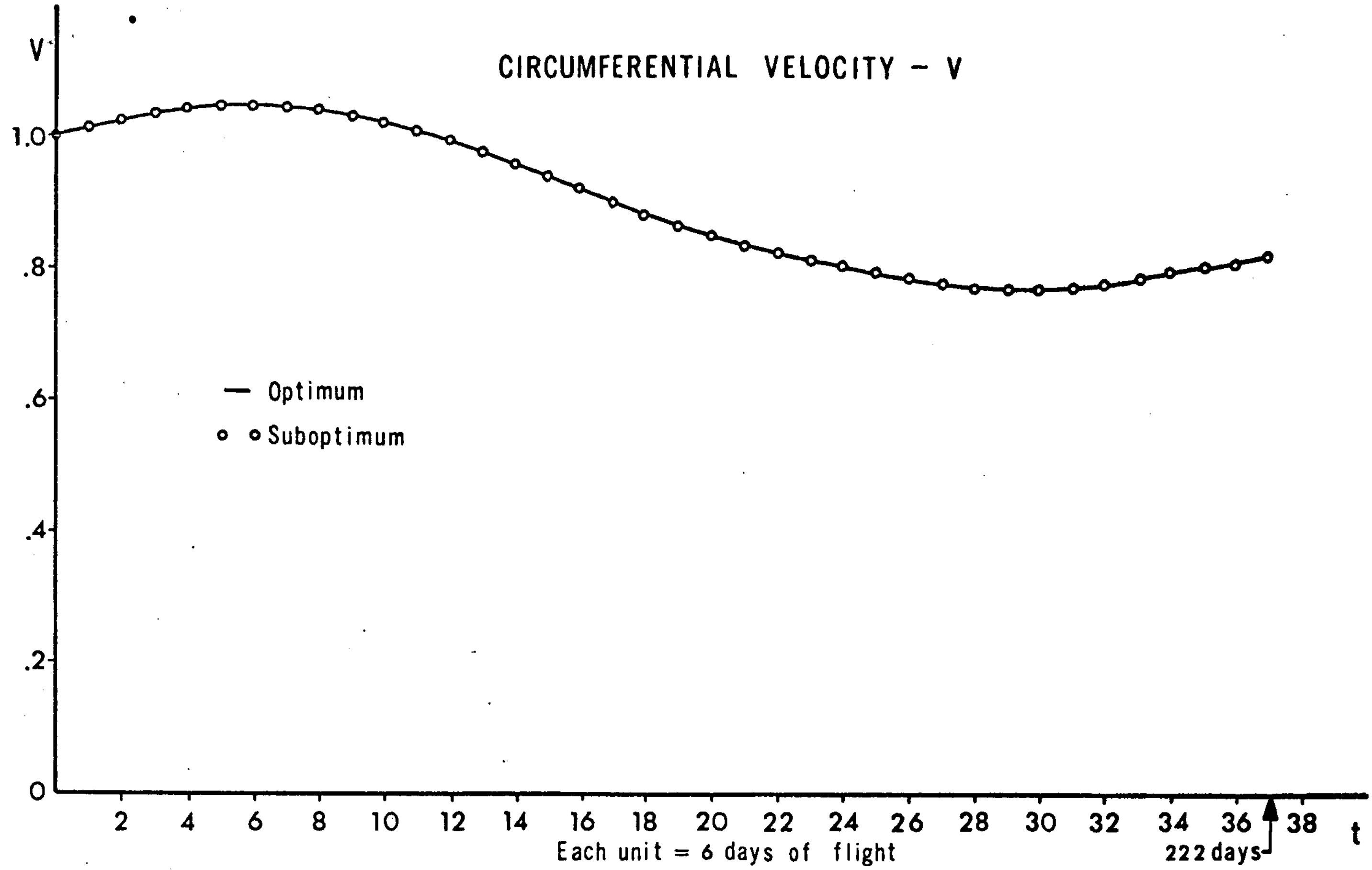

FIGURE 1

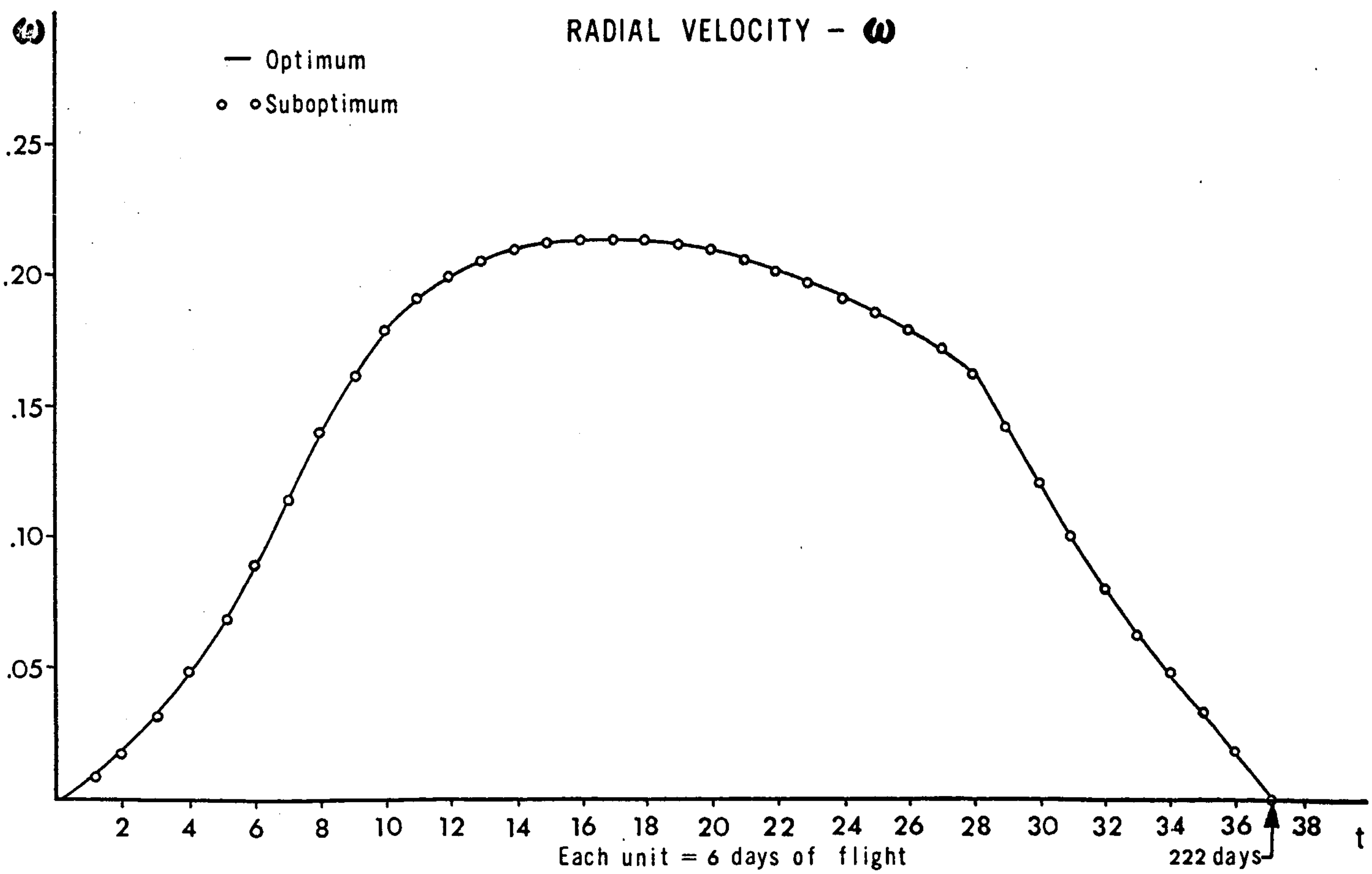

FIGURE 2

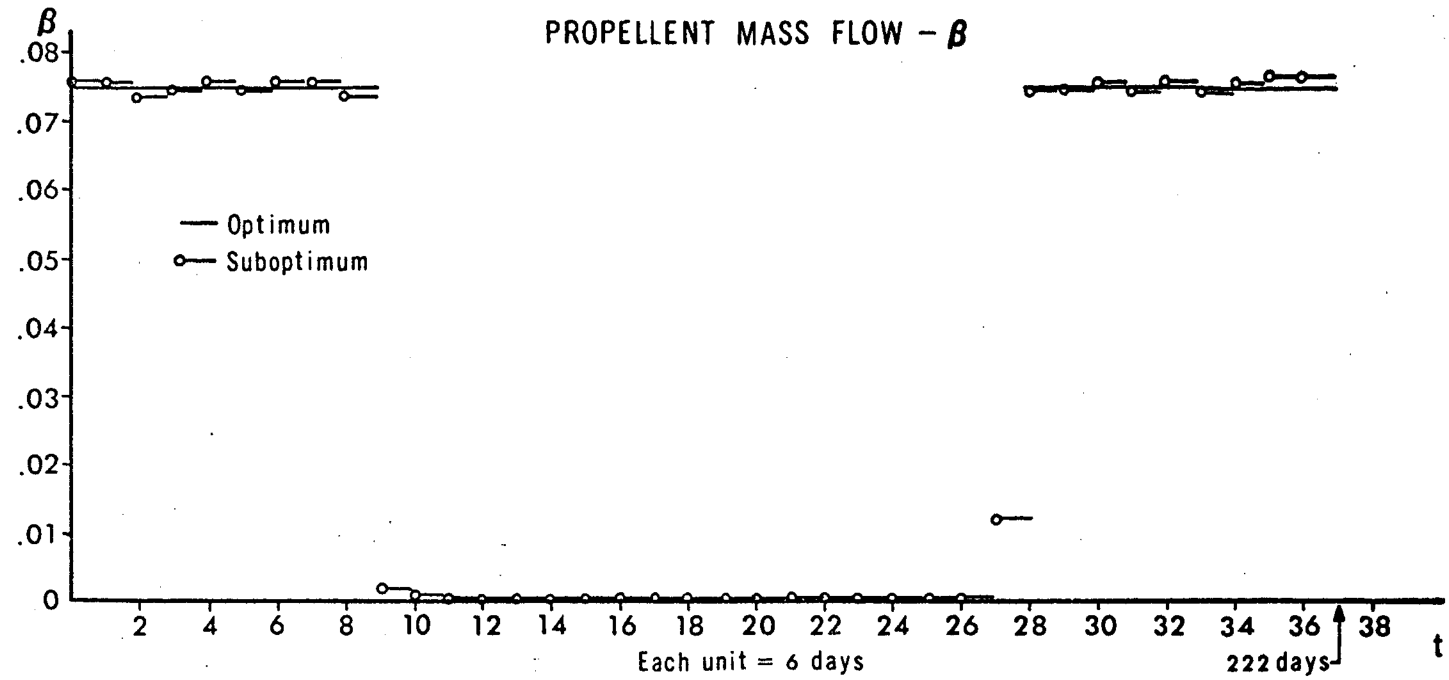

FIGURE 3

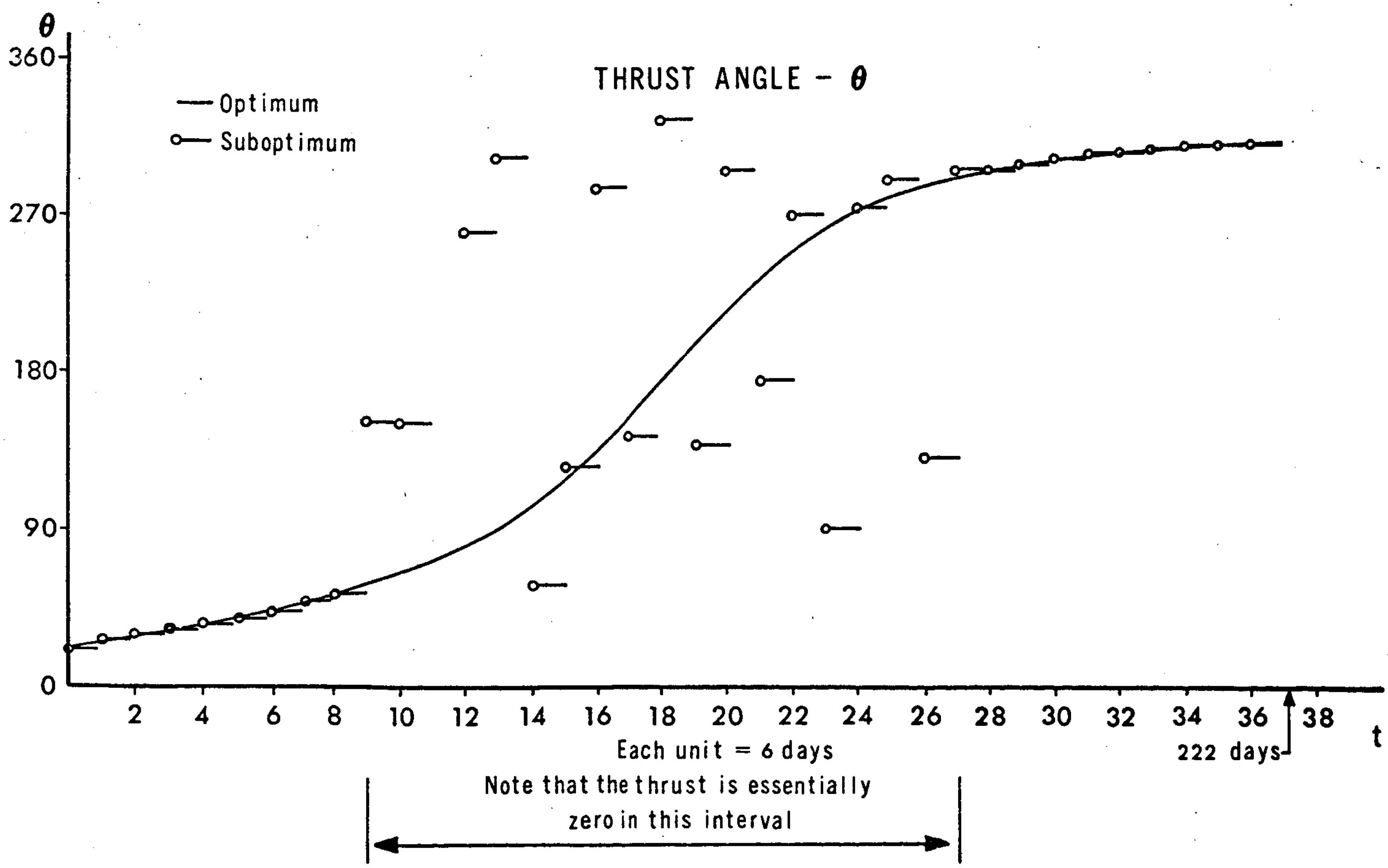

FIGURE 4